Moments and Forces on General Convex Bodies in Hypersonic Flow

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Newtonian impact theory is used to derive differential equations for the moments and forces on convex bodies in hypersonic flow. These equations are solved for a wide range of conditions, to give analytical expressions for the moments and forces in which the constants of integration become parameters to be determined for the particular body. It is found that the force solutions partially duplicate previous work, but no previously comparable expressions exist for the moments. The constants of integration are found either by direct calculation of the Newtonian moments and forces at convenient orientations of the body, or by substituting other known values. This latter procedure forms the basis of a method for extending the validity of Newtonian estimates to a wider range of conditions. Several examples are presented including an estimate of the static stability of the Apollo Command Module.

I. Introduction

THE aerodynamic coefficients of bodies traveling at hypersonic speeds are often estimated using Newtonian impact theory. For some simple shapes these coefficients have been found as analytical expressions of the incidence and yaw, but in most cases they are determined by numerical integration of the pressures over the surface of the body at a particular orientation.

In 1967 Jaslow[1] proposed a new method of obtaining the forces on bodies in Newtonian flow. He found that inherent within Newtonian impact theory there were three independent differential relationships between the forces, and also a relationship between the moments. These force equations for axisymmetric and three-dimensional bodies have been demonstrated[2-4] by the present author to be related to Legendre's and Poisson's equation, respectively, so permitting the wealth of mathematic available on the solution of these equations (in Ref. 5, for example) to be applied to finding the forces on a body according to Newtonian impact theory.

The moment equation found by Jaslow was not sufficient in itself to provide solutions for the moments. In this paper further differential relationships between the forces and moments are obtained. The force equations however are not independent of Jaslow's equations, although their different presentation may simplify the solution for certain classes of bodies. The moment equations form a new set, independent of Jaslow's equations, which are solved to provide analytic expressions for the moments similar to those previously obtained for the forces. Jaslow's moment equation can be used to reduce the number of independent constants of integration, but it is often simpler to determine these constants by substituting known values of the moments into the solution. When these reference values are themselves derived from Newtonian theory, the expressions give a Newtonian estimate of the aerodynamic coefficients. However this estimate can often be improved if the reference values are taken from some more reliable source, such as experimental measurement, or some precise calculation. In effect the estimates are then interpolations between known values, but the interpolation method, being based on Newtonian theory takes some account of the physical processes involved. From its limited use in Ref. 4 and here, this technique appears to have considerable potential. When the constants are fixed using experimental values the accuracy of all the estimates depends upon the accuracy of the particular values used. However as the constants of integration appear in the expression as linear parameters, they can be derived from less reliable data by using standard "least squares error" techniques.

Illustrative examples are presented throughout Secs. III–VIII. In Sec. III a two-dimensional body is examined in detail, in Sec. V the moment of any axisymmetric segment is derived to match the result for the forces given in Ref. 4, in Sec. VII the drag of an oscillating cone is found, and in Sec. VIII the static stability of the Apollo Command Module is considered.

II. Derivation of the Differential Equations

Let the total force coefficient ($\mathbf{C}_F$) and the total moment coefficient ($\mathbf{C}_M$) have components

$$\mathbf{C}_F = \mathbf{v}C_D + \mathbf{l}C_L + \mathbf{s}C_Z \tag{1}$$

$$= \mathbf{e}_1 C_A + \mathbf{e}_2 C_N + \mathbf{e}_3 C_S \tag{2}$$

$$\mathbf{C}_M = -\mathbf{e}_1 C_l - \mathbf{e}_2 C_n - \mathbf{e}_3 C_m \tag{3}$$

where $\mathbf{v}$, $\mathbf{l}$, $\mathbf{s}$, and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors as shown in Fig. 1, representing wind and body axes, respectively. If θ and ϕ are spherical polar coordinates fixed in the body, such that

$$\mathbf{v} = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta\cos\phi - \mathbf{e}_3 \sin\theta\sin\phi \tag{4}$$

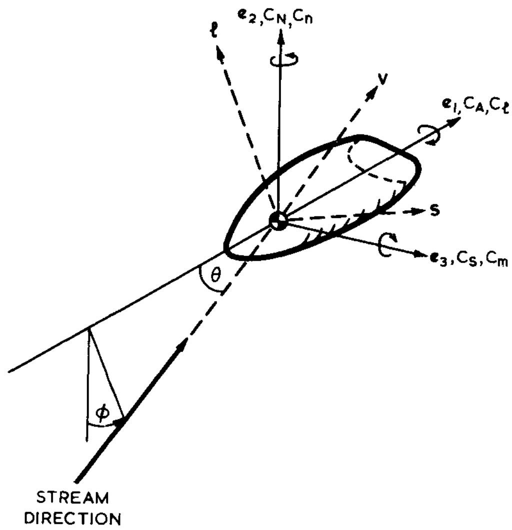

Fig. 1 Axes systems.

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$$\mathbf{l} = -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta \cos\phi - \mathbf{e}_3 \cos\theta \sin\phi \tag{5}$$

$$\mathbf{s} = \mathbf{e}_2 \sin\phi + \mathbf{e}_3 \cos\phi \tag{6}$$

then we have

$$\mathbf{v}_\theta = \mathbf{l} \tag{7}$$

$$\mathbf{l}_\theta = -\mathbf{v} \tag{8}$$

$$\mathbf{s}_\theta = 0 \tag{9}$$

$$\mathbf{v}_\phi = -\mathbf{s}\sin\theta \tag{10}$$

$$\mathbf{l}_\phi = -\mathbf{s}\cos\theta \tag{11}$$

$$\mathbf{s}_\phi = \mathbf{v}\sin\theta + \mathbf{l}\cos\theta \tag{12}$$

where subscripts θ and ϕ denote partial differentiation.

For Newtonian flow the surface pressure coefficient on a surface exposed to the flow is proportional to the square of the component of the freestream velocity normal to the surface, i.e.,

$$\begin{aligned} C_p &= k(\mathbf{v}\cdot\mathbf{n})^2 \quad \text{for} \quad \mathbf{v}\cdot\mathbf{n} \geqq 0 \\ &= 0 \qquad\qquad \text{for} \quad \mathbf{v}\cdot\mathbf{n} \leqq 0 \end{aligned} \tag{13}$$

where $\mathbf{n}$ is a unit vector normal to the surface of the body. Hence the forces and moments of any body in Newtonian flow are given by

$$\mathbf{C}_F = (1/A_R)\int_S C_p \mathbf{n}\, dS = (k/A_R)\int_S (\mathbf{v}\cdot\mathbf{n})^2 \mathbf{n}\, dS \tag{14}$$

$$\mathbf{C}_M = (1/l_R A_R)\int_S C_p (\mathbf{R}\times\mathbf{n})\, dS = (k/l_R A_R)\int_S (\mathbf{v}\cdot\mathbf{n})^2 (\mathbf{R}\times\mathbf{n})\, dS \tag{15}$$

where S is the exposed surface or wetted area of the body (i.e., $\mathbf{v}\cdot\mathbf{n} \geqq 0$) and $\mathbf{R}$ is the radius vector of the surface from the moment center. Following Jaslow,[1] Eq. (15) can be differentiated with respect to θ to give

$$\mathbf{C}_{M\theta} = (2k/l_R A_R)\int_S (\mathbf{v}\cdot\mathbf{n})(\mathbf{v}_\theta\cdot\mathbf{n})(\mathbf{R}\times\mathbf{n})\, dS \tag{16}$$

This differentiation is only valid if the region of integration (S) is fixed or if any change of S with θ produces a negligible contribution to the integral. This latter condition applies in particular when the integrand tends to zero at the boundary. Thus the differentiation is valid in general in our case, for at sharp edges the region of integration does not vary and at rounded edges the integrand tends to zero. It should be noted that the differentiation is not valid near cavities in the body which are partly shielded from the flow. However, for shallow cavities the effect will be small, and for deep cavities (whether partly or fully exposed to the flow) Newtonian theory tends to give misleading results anyway, in this paper we restrict ourselves primarily to convex bodies.

Substituting for $\mathbf{v}_\theta$ in Eq. (16) from Eq. (7) and differentiating again we obtain a new relation

$$\mathbf{C}_{M\theta\theta} = (2k/l_R A_R)\int_S (\mathbf{l}\cdot\mathbf{n})^2(\mathbf{R}\times\mathbf{n})\, dS - (2k/l_R A_R)\int_S (\mathbf{v}\cdot\mathbf{n})^2(\mathbf{R}\times\mathbf{n})\, dS \tag{17}$$

Equations (17) and (15) then give

$$\mathbf{C}_{M\theta\theta} + 4\mathbf{C}_M = (2k/l_R A_R)\int_S \{(\mathbf{l}\cdot\mathbf{n})^2 + (\mathbf{v}\cdot\mathbf{n})^2\}(\mathbf{R}\times\mathbf{n})\, dS \tag{18}$$

In two-dimensional flows, as $\mathbf{v}$ and $\mathbf{l}$ are perpendicular we have

$$(\mathbf{l}\cdot\mathbf{n})^2 + (\mathbf{v}\cdot\mathbf{n})^2 = 1 \tag{19}$$

Equation (18) can then be written

$$\mathbf{C}_{M\theta\theta} + 4\mathbf{C}_M = 2k\mathbf{M}/(l_R A_R) \tag{20}$$

where

$$\mathbf{M}(\theta,\phi) = \int_S (\mathbf{R}\times\mathbf{n})\, dS \tag{21}$$

giving a new differential equation for the moment on a two-dimensional body in Newtonian flow.

A similar expression may be derived for the forces by an identical method giving

$$\mathbf{C}_{F\theta\theta} + 4\mathbf{C}_F = 2k\mathbf{F}/A_R \tag{22}$$

where

$$\mathbf{F}(\theta,\phi) = \int_S \mathbf{n}\, dS \tag{23}$$

It can however be shown that this force equation is contained within those derived by Jaslow for the forces.

For the equations in three dimensions, similar results hold. In three dimensions we have

$$(\mathbf{l}\cdot\mathbf{n})^2 + (\mathbf{v}\cdot\mathbf{n})^2 + (\mathbf{s}\cdot\mathbf{n})^2 = 1 \tag{24}$$

Differentiating $\mathbf{C}_M$ twice with respect to ϕ and using Eqs. (10) and (12) gives

$$\mathbf{C}_{M\phi\phi}\operatorname{cosec}^2\theta + \mathbf{C}_{M\theta}\cot\theta + 2\mathbf{C}_M = (2k/l_R A_R)\int_S (\mathbf{s}\cdot\mathbf{n})^2(\mathbf{R}\times\mathbf{n})\, dS \tag{25}$$

The addition of Eqs. (18) and (25) gives

$$\mathbf{C}_{M\phi\phi}\operatorname{cosec}^2\theta + \mathbf{C}_{M\theta\theta} + \mathbf{C}_{M\theta}\cot\theta + 6\mathbf{C}_M = 2k\mathbf{M}/(l_R A_R) \tag{26}$$

and similarly for the forces

$$\mathbf{C}_{F\phi\phi}\operatorname{cosec}^2\theta + \mathbf{C}_{F\theta\theta} + \mathbf{C}_{F\theta}\cot\theta + 6\mathbf{C}_F = 2k\mathbf{F}/A_R \tag{27}$$

Equations (26) and (27) constitute a set of differential equations for the forces and moments on a body in Newtonian flow. The right-hand sides of the equations are proportional to the terms $\mathbf{M}$ and $\mathbf{F}$ defined by Eqs. (21) and (23). These terms can be seen to represent the moment and force on the body with unit normal pressure acting over the exposed surface S. Hence, from a hydrostatic analogy or otherwise, it is clear that $\mathbf{F}$ and $\mathbf{M}$ depend only on the shape of the boundary of S and not upon the body shape elsewhere. This property is of considerable importance when considering complex bodies with a simple boundary shape, or when comparing a range of bodies with a common boundary.

III. Two-Dimensional Conditions

In two dimensions, only the C_m component of the moment and the C_A and C_N component of the force are of significance. Equations (20) and (22) for these components are

$$C_{A\theta\theta} + 4C_A = 2kF_1/A_R \tag{28}$$

$$C_{N\theta\theta} + 4C_N = 2kF_2/A_R \tag{29}$$

$$C_{m\theta\theta} + 4C_m = 2kM_3/(l_R A_R) \tag{30}$$

The solution to these "linear" differential equations is easily found to be

$$C_A = K_1 \sin 2\theta + K_2 \cos 2\theta + J_1 \tag{31}$$

$$C_N = K_3 \sin 2\theta + K_4 \cos 2\theta + J_2 \tag{32}$$

$$C_m = K_5 \sin 2\theta + K_6 \cos 2\theta + I_3 \tag{33}$$

where K_1 to K_6 are the constants of integration and J_1, J_2, and I_3 are the particular integrals. Thus the evaluation of the moments and forces in two dimensions is reduced to finding the particular integral for the body and then determining the constants K_1 to K_6. A typical evaluation of J_1, J_2, and I_3, and K_1 to K_6 is given by the example which follows. More general methods are discussed later.

Consider the symmetrical circular arc aerofoil shown in Fig. 2. For $\theta \leqq \xi$, the region exposed to the flow PNQ varies with θ, whereas for $\theta \geqq \xi$ the region is constant, being the side NSB. For a constant exposed region both $\mathbf{M}$ and $\mathbf{F}$ will also be constant. Thus from Fig. 2b we see that in this case F_1, the projected area in axial (OB) direction is zero, F_2, the projected area in the normal (OR) direction is NB or "l", and M_3, the

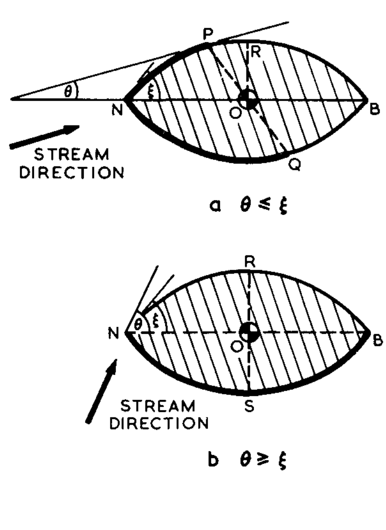

Fig. 2 Biconvex aerofoil.

"constant pressure" moment of the exposed surface about O is zero because O bisects NB.

For a constant $\mathbf{F}$ or $\mathbf{M}$ a particular integral can be found immediately from the differential Eqs. (28–30), giving for the values above $J_1 = I_3 = 0$ and $J_2 = kl/(2A_R)$.

To find a particular integral for $\theta \leq \xi$, when the exposed area varies, the wetted surface PNQ is considered to be replaced by surfaces POQ (Fig. 2a) which have the same boundary points. Then by simple algebra we obtain $F_1 = l(\cos\theta - \cos\xi)/\sin\xi$, $F_2 = l\sin\theta/\sin\xi$, and $M_3 = 0$. Then, as before, putting the values of F_1, F_2, and M_3 in Eqs. (28–30) particular integrals are given by $J_1 = kl(4\cos\theta - 3\cos\xi)/(6A_R \sin\xi)$, $J_2 = 2kl\sin\theta/(3A_R\sin\xi)$, and $I_3 = 0$.

Evaluating K_5 and K_6 in Eq. (33) then gives C_m for any symmetrical circular arc aerofoil at any incidence. As the aerofoil has the same upper and lower surface, C_m will be an odd function of θ which requires $K_6 = 0$. Hence with $I_3 = 0$ also, the moment coefficient is found from Eq. (33) to be simply

$$C_m = \tfrac{1}{2} C_{m\theta}(0) \sin 2\theta \tag{34}$$

where $C_{m\theta}(0)$ is found by calculation to be $l^2 k \sin 2\xi/(3 l_R A_R)$.

The particular integrals that have been found for the forces are discontinuous at $\theta = \xi$. To compensate for this change in the particular integral, the "constants" of integration also change at $\theta = \xi$, and we need two different sets of constants, given by

$$K_n = K_n^- \quad \theta \leq \xi \tag{35}$$

$$K_n = K_n^+ \quad \theta \geq \xi \tag{36}$$

For the problem in hand, symmetry conditions establish

$$K_1^- = K_2^+ = K_3^+ = K_4^- = 0$$

The remaining four constants can be found in various ways. The method employed here is to require that $C_A(0)$ and $C_N(\pi/2)$ are as established by straightforward calculation, viz

$$(8 - 9\cos\xi + \cos 3\xi) kl/(12 A_R \sin\xi)$$

and

$$(2 + \cos^2\xi) kl/(3A_R)$$

respectively, and that C_A, C_N are continuous at $\theta = \xi$. The results are

$$K_1^+ = kl(1 - \cos 2\xi)/(6A_R) \tag{37}$$

$$K_2^- = -kl(2 - \cos 2\xi)/(6A_R \tan\xi) \tag{38}$$

$$K_3^- = -kl(1 - \cos 2\xi)/(6A_R \tan\xi) \tag{39}$$

$$K_4^+ = -kl(2 + \cos 2\xi)/(6A_R) \tag{40}$$

which on substitution in Eqs. (31) and (32) give the force coefficients on any symmetrical circular arc aerofoil, that is

$$C_A = kl[4\cos\theta - 3\cos\xi - (2 - \cos 2\xi)\cos\xi\cos 2\theta]/(6A_R \sin\xi) \quad \theta \leq \xi \tag{41}$$

$$= kl\sin^2\xi \sin 2\theta/(3A_R) \quad \theta \geq \xi \tag{42}$$

$$C_N = 2kl\sin\theta(1 - \cos^3\xi\cos\theta)/(3A_R\sin\xi) \quad \theta \leq \xi \tag{43}$$

$$= kl[3 - (2 + \cos 2\xi)\cos 2\theta]/(6A_R) \quad \theta \geq \xi \tag{44}$$

IV. General Solution

The three-dimensional equations (26) and (27) can be recognized as reduced forms of Poisson's equation with a complementary function given by

$$C \cdot F = \sum_{m=0}^{2} (\mathbf{A}_m \cos m\phi + \mathbf{B}_m \sin m\phi) P_2{}^m(\cos\theta) + (\mathbf{a}_m \cos m\phi + \mathbf{b}_m \sin m\phi) Q_2{}^m(\cos\theta) \tag{45}$$

where the Associated Legendre Functions of the first and second kinds are given by $P_2{}^0 = \frac{1}{2}(3\cos^2\theta - 1)$, $P_2{}^1 = 3\sin\theta\cos\theta$, $P_2{}^2 = 3\sin^2\theta$, $Q_2{}^0 = P_2{}^0 \log\cot(\theta/2) - 1.5\cos\theta$, $Q_2{}^1 = -dQ_2{}^0/d\theta$, and $Q_2{}^2 = \sin\theta d(\operatorname{cosec}\theta Q_2{}^1)/d\theta$. The functions $Q_2{}^m$ have logarithmic singularities at $\theta = 0$ and $\theta = \pi/2$, so that in most cases $\mathbf{a}_m$ and $\mathbf{b}_m$ can be excluded from the solution. Should it be necessary to include them (e.g. Ref. 3), then they can be manipulated by an extension of the same techniques as are used here.

The constants of integration $\mathbf{A}_m$ and $\mathbf{B}_m$ (and $\mathbf{a}_m$ and $\mathbf{b}_m$, where appropriate) may be obtained by substituting into the full solution known or calculated values of the moments and forces at particular orientations, as demonstrated for the biconvex aerofoil of the previous section. Nearly all bodies of aerodynamic interest are symmetrical about a plane which we may define at $\phi = 0$. This symmetry requires that C_A, C_N, and C_m are even functions of ϕ, while C_l, C_n, and C_S are odd functions; thus in this case, the moments and forces can be written simply as

$$C_l = B_{11}\sin\phi P_2{}^1 + B_{21}\sin 2\phi P_2{}^2 + I_l \tag{46}$$

$$C_n = B_{12}\sin\phi P_2{}^1 + B_{22}\sin 2\phi P_2{}^2 + I_n \tag{47}$$

$$C_m = A_{03}P_2{}^0 + A_{13}\cos\phi P_2{}^1 + A_{23}\cos 2\phi P_2{}^2 + I_m \tag{48}$$

$$C_A = A_{01}P_2{}^0 + A_{11}\cos\phi P_2{}^1 + A_{21}\cos 2\phi P_2{}^2 + J_A \tag{49}$$

$$C_N = A_{02}P_2{}^0 + A_{12}\cos\phi P_2{}^1 + A_{22}\cos 2\phi P_2{}^2 + J_N \tag{50}$$

$$C_S = B_{13}\sin\phi P_2{}^1 + B_{23}\sin 2\phi P_2{}^2 + J_S \tag{51}$$

We observe that the moments have 7 constants of integration and the forces have 8 constants. However, not all of these constants of integration are independent, for other relations between the force and moment equations exist.[1] These relationships are most easily exploited by substituting $C_l(\theta, \phi)$ etc. from Eqs. (46–51) in the following equations:

$$C_{A\theta}(0,0) = (2k/A_R)\int_S (\mathbf{e}_1 \cdot \mathbf{n})^2(\mathbf{e}_2 \cdot \mathbf{n})\, dS = 2C_N(0,0) \tag{52}$$

$$C_{N\theta}(0,0) = (2k/A_R)\int_S (\mathbf{e}_2 \cdot \mathbf{n})^2(\mathbf{e}_1 \cdot \mathbf{n})\, dS = 2C_A(90,0) \tag{53}$$

$$C_{S\phi}(90,90) = (2k\sin\theta/A_R)\int_S (\mathbf{e}_3 \cdot \mathbf{n})^2(\mathbf{e}_2 \cdot \mathbf{n})\, dS = 2C_N(90,90) \tag{54}$$

$$C_{S\theta}(0,90) = (2k/A_R)\int_S (\mathbf{e}_3 \cdot \mathbf{n})^2(\mathbf{e}_1 \cdot \mathbf{n})\, dS = 2C_A(90,90) \tag{55}$$

$$2C_m(90,90) + C_{l\theta}(90,90) + C_{n\phi}(90,90) = (2/l_R A_R)\int (\mathbf{e}_3 \cdot \mathbf{n}) \sum_{r=1}^{3} (\mathbf{e}_r \cdot \mathbf{n})(\mathbf{e}_r \cdot \mathbf{R} \times \mathbf{n})\, dS = 0 \tag{56}$$

to give

$$3A_{11} - 2A_{02} = 2J_N(0,0) - J_{A\theta}(0,0) \tag{57}$$

$$A_{01} + 3A_{12} - 6A_{21} = 2J_A(90,0) - J_{N\theta}(0,0) \tag{58}$$

$$A_{02} + 6A_{22} - 6B_{23} = 2J_N(90,90) - J_{S\phi}(90,90) \tag{59}$$

$$A_{01} + 6A_{21} - 3B_{13} = 2J_A(90,90) - J_{S\theta}(90,90) \tag{60}$$

$$A_{03} + 6A_{23} + 3B_{11} + 6B_{22} = 2I_m(90,90) + I_{n\theta}(90,90) + I_{l\phi}(90,90) \tag{61}$$

These equations reduce the number of independent constants of integration to 6 for the moments and 4 for the forces. This result for the forces has been found previously.[4] The particular integrals $\mathbf{I}$ and $\mathbf{J}$ are obtained by solving the differential Eqs. (26) and (27) with the values of $\mathbf{M}$ and $\mathbf{F}$ for the particular body. However, we can use the fact that $\mathbf{M}$ and $\mathbf{F}$ depend only on the shape of the boundary $\mathbf{v} \cdot \mathbf{n} = 0$ and not upon the body shape elsewhere to find some particular integrals which apply to a wide class of shapes.

V. Some Common Special Cases

Consider first any body exposing a constant surface to the flow for some range of θ and ϕ (e.g., Fig. 3). Then in this case as the boundary does not change with θ or ϕ, $\mathbf{M}$ and $\mathbf{F}$ are constant, and from Eqs. (26) and (27) we find

$$\mathbf{I} = k\mathbf{M}/(3l_R A_R) \tag{62}$$

$$\mathbf{J} = k\mathbf{F}/(3A_R) \tag{63}$$

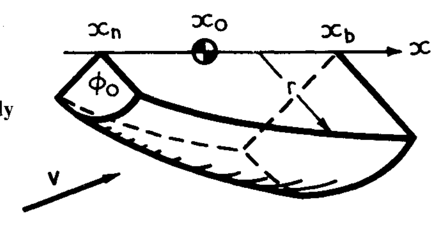

Fig. 3 Segment of body of revolution.

The forces but not the moments on such a body have been considered previously.[4]

For the moments from Eqs. (46–48) (with symmetry about $\phi = 0$ giving $I_l = \mathbf{M}\cdot\mathbf{e}_1 = 0$ and $I_n = \mathbf{M}\cdot\mathbf{e}_2 = 0$)

$$C_l = B_{11}\sin\phi P_2{}^1 + B_{21}\sin 2\phi P_2{}^2 \qquad (64)$$

$$C_n = B_{12}\sin\phi P_2{}^1 + B_{22}\sin 2\phi P_2{}^2 \qquad (65)$$

$$C_m = A_{03}P_2{}^0 + A_{13}\cos\phi P_2{}^1 + A_{23}\cos 2\phi P_2{}^2 + kM_3/(3l_R A_R) \qquad (66)$$

where M_3 is $\mathbf{M}\cdot\mathbf{e}_3$, and Eq. (61) reduces to

$$3B_{11} + 6B_{22} + A_{03} + 6A_{23} = 2kM_3/(3l_R A_R) \qquad (67)$$

Thus the Newtonian aerodynamic moments of any (symmetrical) body which shows a constant surface to the flow are obtained simply by finding the values of the independent constants in Eqs. (64–67). As a particular example, we consider any segment of an axisymmetric body such as that shown in Fig. 3. For these bodies, the forces have been derived[4] in terms of C_{DO} and C_{LO} only. For the moments we require (at $\theta = \phi = 0$)

$$C_{m0} = -\frac{2k\sin(\frac{1}{2}\phi_0)}{l_R A_R}\int_{x_n}^{x_b}\frac{(x + r\,dr/dx - x_0)r(dr/dx)^2 dx}{1 + (dr/dx)^2} \qquad (68)$$

$$C_{m\theta 0} = \frac{k(\phi_0 + \sin\phi_0)}{l_R A_R}\int_{x_n}^{x_b}\frac{(x + r\,dr/dx - x_0)r(dr/dx)\,dx}{1 + (dr/dx)^2} \qquad (69)$$

where M_3 is the constant

$$M_3 = -2\sin(\tfrac{1}{2}\phi_0)\int_{x_n}^{x_b}(x + r\,dr/dx - x_0)r\,dx \qquad (70)$$

Then the moments are given by Eqs. (64–66) with

$$B_{11} = B_{21} = 0,\quad 3B_{12} = C_{m\theta 0}(\phi_0 - \sin\phi_0)/(\phi_0 + \sin\phi_0)$$

$$B_{22} = (1 - \cos\phi_0)[kM_3/(l_R A_R) - C_{m0}]/18$$

$$A_{03} = -kM_3/(3l_R A_R) + C_{m0},\quad A_{13} = \tfrac{1}{3}C_{m\theta 0}$$

and

$$A_{23} = [kM_3/(l_R A_R) - C_{m0}](2 + \cos\phi_0)/18$$

For Eqs. (64–67) to give the moments on a body it is not necessary for the boundary to be fixed on the body, so long as its shape remains constant. An example of this is shown in Fig. 4, which depicts a part conical nose projecting from a sphere. Then for $\theta \leq \theta_c$ the boundary is a circle and $\mathbf{M}$ is again constant.

Another extensive class of bodies for which Eqs. (64–67) also apply are bodies which are "centrally" symmetric. That is bodies for which there exists a position for the origin such that if $P(x, y, z)$ is a surface point then so is $P(-x, -y, -z)$ (see Fig. 5). With such bodies, if P is exposed to the flow then P' will not be, and vice versa. Thus the contribution to $\mathbf{M}$ about the origin from two similar surface elements centered on P and P' is the same irrespective of the orientation of the body. For the whole surface this result implies that about the origin $\mathbf{M}$ is a constant, and Eqs. (64–67) again apply.

VI. Simplification with Two or More Planes of Symmetry

Before presenting further examples relating to the particular integrals discussed previously, consider bodies with planes of symmetry at $\phi = r\pi/n$, where n is the number of planes of symmetry and r takes all integer values between 0 and $n-1$. When $n = 2$ we find that applying symmetry conditions at $\phi = 0$ and $\pi/2$ to Eqs. (46–48) gives B_{11}, B_{22}, A_{03}, and A_{23}, all to be zero. Equations (46–48) then reduce to

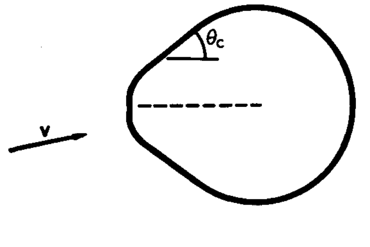

Fig. 4 Body with spherical base.

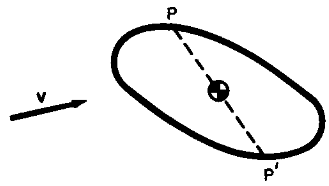

Fig. 5 A centrally symmetric body.

$$C_l = B_{21}\sin 2\phi P_2{}^2 + I_l \qquad (71)$$

$$C_n = B_{12}\sin\phi P_2{}^1 + I_n \qquad (72)$$

$$C_m = A_{13}\cos\phi P_2{}^1 + I_m \qquad (73)$$

with only three constants on integration, where it has been assumed in these and similar succeeding equations that the particular integral terms themselves exhibit the correct form of symmetry. It can be shown also that the force equations for these conditions have only two independent constants of integration.[4]

When $n = 3$ we have, similarly,

$$C_l = I_l \qquad (74)$$

$$C_n = B_{12}\sin\phi P_2{}^1 + B_{22}\sin 2\phi P_2{}^2 + I_n \qquad (75)$$

$$C_m = -B_{12}\cos\phi P_2{}^1 + B_{22}\cos 2\phi P_2{}^2 + I_m \qquad (76)$$

and for $n > 3$ we find

$$C_l = I_l \qquad (77)$$

$$C_n = B_{12}\sin\phi P_2{}^1 + I_n \qquad (78)$$

$$C_m = -B_{12}\cos\phi P_2{}^1 + I_m \qquad (79)$$

In particular for axisymmetric bodies ($n \to \infty$) there is no moment about the axis of symmetry and thus Eq. (77) becomes $C_l = I_l = 0$. Further the ϕ dependence can be removed from Eqs. (78) and (79) by writing

$$C_M = -B_{12}P_2{}^1 + I \qquad (80)$$

where C_M is taken to be perpendicular to a plane containing $\mathbf{v}$ and $\mathbf{e}_1$. Consider now an axisymmetric body which has a constant exposed surface to the stream, or is centrally symmetric (which is equivalent to fore and aft symmetric for an axisymmetric body), then for such bodies $I = 0$ and C_M is given from Eq. (80) by

$$C_M = \tfrac{1}{2}C_{M\theta}(0)\sin 2\theta \qquad (81)$$

This result is the same as that of Eq. (34) for two-dimensional bodies. The value of $C_{M\theta}(0)$ for a body $r = r(x)$ where x_n, x_m, and x_0 are the x coordinates of the nose, the point of maximum diameter, and the center of moment, respectively, can be obtained from

$$C_{m\theta}(0) = -\frac{2k\pi}{l_R A_R}\int_{x_n}^{x_m}\frac{(dr/dx)(x - x_0 + r\,dr/dx)r\,dx}{1 + (dr/dx)^2} \qquad (82)$$

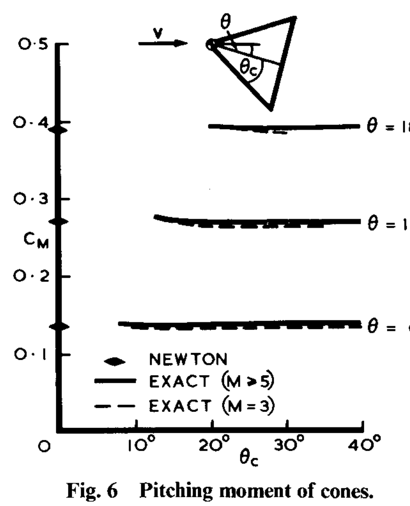

Fig. 6 Pitching moment of cones.

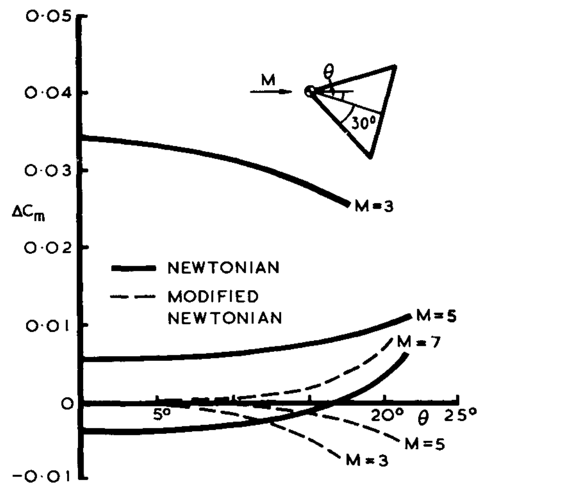

Fig. 7 Error in moment coefficient for a 30° cone at $M = 3$, 5, and 7.

We consider as an example a right circular cone at incidence, for which exact values of the forces have been obtained[6,7] by numerical integration of the flow equation. In Ref. 7 moment values are given also, but unfortunately these are in error. Here the correct values for C_M about the nose with the cone length as reference length are obtained from C_N by using the exact relationship

$$C_M = -\tfrac{2}{3}C_N \sec^2\theta_c \qquad (83)$$

To derive the Newtonian prediction for the moment of a right circular cone at an incidence less than its semiapex angle, we put $R = x\tan\theta_c$ in Eq. (82) and substitute for $C_{M\theta}(0)$ in Eq. (81) to obtain

$$C_M = -\tfrac{1}{3}k\sin 2\theta \qquad (84)$$

This Newtonian estimate suggests that the moment coefficient is independent of the cone semiapex angle. In Fig. 6 exact values of C_M are plotted against θ_c for a range of incidence and Mach number. We observe that the variation of C_M with θ_c is indeed small, and further than when $k = 2$, Eq. (84) provides a close approximation to C_M. This is shown in more detail in Fig. 7 by plotting $C_M = C_M$ (Newtonian)/C_M(exact)-1 for a cone with a 30° semiapex angle. At the higher Mach numbers ΔC_M is very small, but for $M = 3$ significant differences occur between the exact and estimated values of C_M. Using exact values for $C_{M\theta}(0)$ from Ref. 6 in Eq. (81) improves the C_M estimate for the lower Mach numbers, as is indicated by the broken lines in Fig. 7.

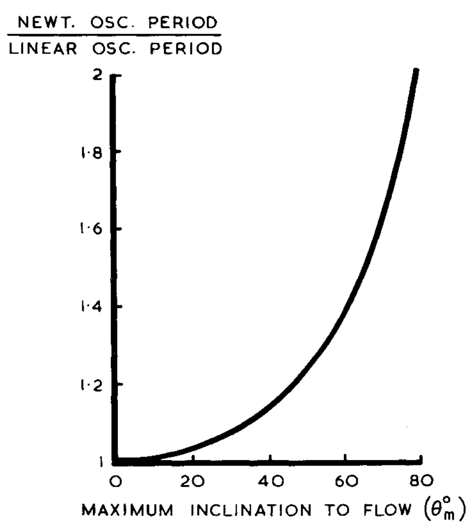

Fig. 8 Period increase from a Newtonian nonsinusoidal oscillation.

The forces on the cone are easily found from the present analysis or from Ref. 2 to be

$$C_N = \tfrac{1}{2}k\cos^2\theta_c \sin 2\theta \qquad (85)$$

$$C_A = \tfrac{1}{2}k\cos^2\theta_c + \tfrac{1}{2}k(2-3\cos^2\theta_c)\cos^2\theta \qquad (86)$$

These forces are used to find the drag of an oscillating cone in Sec. VII.

VII. Oscillating Bodies with Nonlinear Moments

We consider two-dimensional or axisymmetric bodies which have either a constant exposed area or are centrally symmetric. The moments of such bodies are given by Eqs. (34) and (81) to be proportional to $\sin\theta\cos\theta$. If we assume that the velocity of the stream is much larger than the local body velocities caused by the forces and moments (i.e., the translation velocities and the damping are small), then the angular acceleration of the body $\ddot\theta$ depends on the ratio of the moment to the body inertia and is given by

$$\mathscr{M}\ddot\theta + \dot\theta_0{}^2 \sin\theta\cos\theta = 0 \qquad (87)$$

where $\mathscr{M} = -I_e\dot\theta_0{}^2/[C_{m\theta}(0)l_R A_R q]$ and $\dot\theta_0$ is the angular velocity at $\theta = 0$. Integrating this equation gives the angular velocity at any θ as

$$\dot\theta = \dot\theta_0(1-\mathscr{M}^{-1}\sin^2\theta)^{1/2} \qquad (88)$$

The maximum inclination of the body to the flow θ_m is given by $\dot\theta = 0$, that is

$$\sin\theta_m = \mathscr{M}^{1/2} \qquad (89)$$

The oscillation period when $0 < \mathscr{M} < 1$ is given by further integration to be

$$T = \frac{4\sin\theta_m}{\dot\theta_0}K(\sin\theta_m) \qquad (90)$$

where K is the complete elliptic integral of the first kind. Note that for $\mathscr{M} > 1$ or $\mathscr{M} < 0$ the body rotates with period

$$\begin{aligned} T &= (4/\dot\theta_0)K(\mathscr{M}^{-1/2}) & \mathscr{M} > 1 \\ &= \frac{4}{\dot\theta_0}\left(\frac{\mathscr{M}}{1-\mathscr{M}}\right)^{1/2} K[(1-\mathscr{M})^{-1/2}] & \mathscr{M} < 0 \end{aligned} \qquad (91)$$

The oscillation period of Eq. (90) is greater than that using the usual linear assumption (i.e., $C_{m\theta} =$ const). In Fig. 8 the increase in the period is shown to be significant for θ_m greater than about 20°. The inclusion of damping terms gives an equation which is not analytically solvable. However, the gross features of the motion can be described using the approximate method of Ref. 8.

One effect of an oscillating body is to increase the drag above its minimum drag orientation. The time average forces on the body are given by

$$\bar{C}_F = \frac{1}{T}\int_0^T C_F\,dt = \frac{4}{T}\int_0^{\max(\theta_m,\pi/2)} \frac{C_F}{\dot\theta}\,d\theta \qquad (92)$$

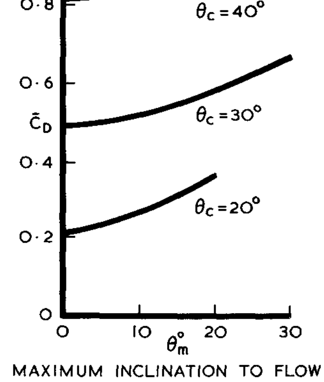

Fig. 9 Drag of an oscillating cone.

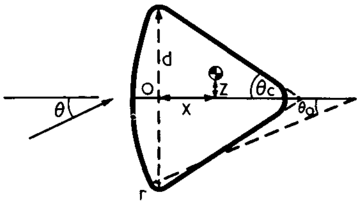

Fig. 10 Shape resembling Apollo Command Module.

For a cone which is oscillating with $\theta_m < \theta_c$, this becomes

$$\bar{C}_D = \frac{2k}{T\theta_0}\int_0^{\theta_m} \frac{3\sin^2\theta\cos\theta\cos^2\theta_c + 2\cos^3\theta\sin^2\theta_c}{(1-\sin^2\theta/\sin^2\theta_m)^{1/2}}\, d\theta \tag{93}$$

$$= \frac{\pi k}{8K(\sin\theta_m)}[4\sin^2\theta_c + (3-5\sin^2\theta_c)\sin^2\theta_m] \tag{94}$$

The average drag of oscillating cones with $\theta_c = 20°, 30°, 40°$, is shown in Fig. 9.

VIII. Static Stability of the Apollo Command Module

As a final example, consider the shape illustrated in Fig. 10, which, when $r/d = 0.05, \theta_0 = 23°$, and $\theta_c = 33°$ is the basic "clean" shape of the Apollo Command Module. In Ref. 9 the static stability conditions of Apollo are obtained from extensive wind-tunnel measurements over a range of Mach numbers. Here we evaluate the modified Newtonian technique by estimating the same static stability results using only two experimental measurements at each Mach number, namely C_A at $\theta = 0$ and C_M at $\theta = 30°$.

To obtain the moments and forces according to Newtonian theory, we observe first that for $\theta \leq \theta_c$, the region of the body exposed to the flow does not include any of the rear cone portion. Thus in this case, the exposed portion of the body may be considered as part of a body which is centrally symmetric about the point marked 0 in Fig. 10. We have already shown that for centrally symmetric bodies the moment is proportional to $\sin\theta\cos\theta$ about 0, and about any other point $P(X, Z)$ we have

$$C_M = C_0 \sin\theta\cos\theta + (Z/d)C_A - (X/d)C_N \tag{95}$$

where C_0 is a constant whose value is given by $[C_{M\theta}(0) - (X/d)C_{N\theta}(0)]$. The forces C_A and C_N can be obtained from Eqs. (49) and (50) or otherwise[4] to be

$$C_A = \tfrac{1}{2}(k - C_{A0} - kr/d) + k(r/d)\cos\theta + \tfrac{1}{2}(3C_{A0} - k - kr/d)\cos^2\theta \tag{96}$$

$$C_N = k(r/d)\sin\theta + (k - C_{A0} - kr/d)\sin\theta\cos\theta \tag{97}$$

where C_{A0} is found by direct calculation to be given by

$$C_{A0} = \tfrac{1}{2}k(1 - 2r/d + 2(r/d)\sin\theta_0)^2(2 - \sin^2\theta_0) + \tfrac{4}{3}k(r/d)(1 - 2r/d)(2 - 3\sin\theta_0 + \sin^3\theta_0) + 2k(r/d)^2\cos^4\theta_0 \tag{98}$$

and the reference area is $\pi d^2/4$.

Using the experimental value[9] of C_{A0} in Eqs. (96–98) enables C_A and C_N to be determined as functions of θ. In Fig. 11 the values of C_A and C_N found by this means are compared with experimental values[9] at $M = 5$. The excellent agreement shown is typical of Mach numbers greater than 3.

To determine C_M from Eq. (95), a value of C_0 is required. As $C_{M\theta}(0)$ (which occurs in the expression for C_0) is difficult to determine from experimental measurements, it is easier to find C_0 by substituting an experimental value of C_M with $\theta \neq 0$ direct into Eq. (95). Also shown in Fig. 11 is C_M from Eq. (95) with $X/d = 0.13$, $Z/d = 0.035$, and C_0 determined from $C_M(30°)$. We see that the trim incidence is predicted to be $22\frac{1}{2}°$ at $M = 5$ for this center of gravity position. Once the trim incidence is determined the forces may be obtained from Eqs. (96) and (97). At $M = 5$ these are found to be $C_A = 1.29$ and $C_N = -0.08$, as shown in Fig. 11.

Thus it is demonstrated how the method proposed here may be used to provide a rapid means of evaluating vehicle performance from the minimum of tunnel data. The analytical form of the estimate makes the evaluation of the gradients of the forces and moments ($C_{M\theta}$, for example) straightforward, by differentiating the appropriate equation to obtain an analytic representation of the gradient.

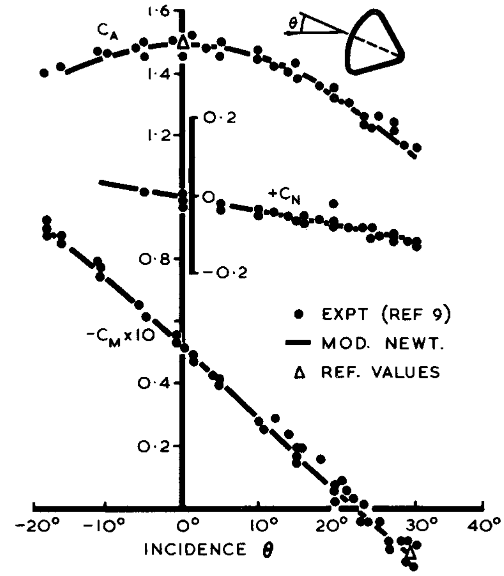

Fig. 11 Apollo Command Module at $M = 5$.

IX. Conclusions

The moments and forces of convex bodies in Newtonian flow may be considered as solutions of Poisson's equation. Using mathematics pertaining to Poisson's equation,[5] the complementary function and a wide range of particular integrals are found which depend exclusively on the boundary of that region of the body exposed to the flow. The method provides a powerful new tool for obtaining general analytic expressions for the Newtonian aerodynamic coefficients of a wide range of practical body shapes over a range of incidence and yaw.

The constants of integration appear as parameters in these expressions which are determined by reference to the detailed body shape. One method of determining these parameters is to substitute into the expressions known values of the moments and forces with their appropriate attitudes. If Newtonian theory values are used throughout, the often complex expressions for the moments and forces are derived more simply and in a more revealing form.

The parameters in the expressions can also be determined using known values of the moments and forces not derived from Newtonian theory. This technique provides an extension to the customary modification of Newtonian theory in which k is chosen to match experimental data. In the present approach, any or all of the constants of integration can be chosen on such a basis to match wind-tunnel results or other accurately computed values. In all the examples examined so far, the effect of this increased freedom is to provide a reliable estimate of the aerodynamic coefficients over a wide range of attitude and M from relatively few experimental measurements.

An advantage of the moments and forces being represented as analytic expressions is the ease with which they undergo further analysis. Gradients may be obtained simply by differentiating the expressions, and other processes (such as optimization) may be advanced analytically so as to reduce the amount of computing which remains.

An example in Sec. VII deals with the oscillations and rotations of a body with negligible damping. When the motion of the body under the action of the nonlinear forces and moments prescribed by Newtonian theory is compared with the usual linear assumption, it is found that for large oscillations (e.g., $\theta_{max} > 20°$) the period of the motion is significantly increased.

A further example in Sec. VIII uses the modified Newtonian theory technique to investigate the static stability of the Apollo Command Module. Using tunnel results from $\theta = 0$ and $\theta = 30^\circ$ only, the static stability conditions of the module are derived and are found to be close to those obtained using the full range of experimental results.

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Velocity Distribution Functions Near the Leading Edge of a Flat Plate

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The electron beam fluorescence technique has been used to determine the velocity distribution function within several mean free paths of the leading edge in a rarefied hypersonic helium flow. The Doppler shift and broadening of the 5016 Å He line was measured with a computer controlled Fabry-Perot interferometer. For analysis, the velocity distribution function was approximated with a bimodal model consisting of two Gaussian functions, which was convoluted with the appropriate instrument function and natural line profile and then fitted to the experimental data. Macroscopic properties in the vicinity of the leading edge, slip velocities, and temperature jumps have been obtained from the complete velocity distribution functions.

Nomenclature

A	= height of distribution function
c	= speed of light
D	= orifice diameter
H	= enthalpy
k	= Boltzmann constant
$L(\lambda)$	= natural line shape of the He $3^1P - 2^1S$ transition
M	= Mach number
$M(\lambda, T)$	= Maxwellian distribution function
$\dot{m}$	= mass flow
m	= atomic mass of helium
n	= weighting of distribution function
p	= pressure
R	= gas constant
S	= speed ratio
$S(\lambda)$	= spectral profile
T	= temperature
u	= particle velocity
U	= mean velocity
ΔU	= $U_1 - U_2$
x	= axial distance from leading edge
y	= height above flat plate
z	= direction normal to flow in plane of plate surface
α	= energy accommodation coefficient
β	= bevel angle
λ	= mean free path
λ	= wavelength
λ_0	= He 5016 Å wavelength
$\Delta\lambda$	= deviation $\lambda - \lambda_0$
η	= coefficient
μ	= viscosity
ρ	= density
σ	= tangential momentum accommodation coefficient

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Index category: Rarefied Flows.

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